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Linear feedback control for convection-diffusion

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Abstract *This study is concerned with linear feedback control in finite element modeling of transient convection-diffusion in an incompressible fluid. Our aim is to introduce a linear feedback that solves the corresponding tracking problem for any sufficiently smooth target temperature field. The control aspects are discussed and we summarize convergence results. Numerical results for a test problem are included to illustrate the approach.*

Introduction

The use of feedback control in engineering simulation is increasing (Gunzburger and Kim, 1998; Gunzburger and Manservisi, 2000a, b; Gunzburger *et al.*, 2000) and the ideas are also related to dynamic grid adaption based on computational error control. Gunzburger and Manservisi (2000a) consider the tracking problem for the Navier-Stokes equations that govern motion of a fluid for the incompressible case. In the present work, we consider the following initial-boundary value problem

$$\begin{aligned} &T_t + \mathbf{u}\cdot\nabla T = \alpha\nabla^2 T + g, \quad \text{in}\,\Omega \\ &T|_{\partial\Omega} = 0, \quad \mathrm{T}(\cdot,0) = T_0 \end{aligned} \tag{1}$$

where $\Omega \subset \mathscr{R}^n$ $(n = 2, 3)$ is a bounded domain. In equation (1) T is the temperature, $\mathbf{u}$ is the convective velocity, α is the thermal diffusivity, g is a source term and T_0 is the initial temperature, which satisfies $T_0|_{\partial\Omega} = 0$. The velocity field satisfies the incompressibility condition $\nabla\cdot\mathbf{u} = 0$. The tracking problem requires finding a suitable heat source function g (depending on both T and T_*) such that the solution of equation (1) approaches for large t, a desired target temperature profile T_* with $T_*|_{\partial\Omega} = 0$, in some norm.

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The controller we propose for the convection-diffusion equation and the subsequent analysis follow closely the treatment by Gunzburger and Manservisi (2000a) and in some sense complement that study since together they provide a comprehensive linear feedback control law for solving the tracking problem for both velocity and temperature.

Control theoretical aspects

The simplest tracking problem encountered in applications is that of tracking a zero target signal. Consider the following linear time invariant (LTI) system in state-space form:

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{v} \\ \mathbf{y} &= C\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{2}$$

where A, B and C are constant real matrices of size $n \times n$, $n \times m$ and $p \times n$, respectively. Here $\mathbf{x}(t)$ is the state vector (size $n \times 1$), $\mathbf{y}(t)$ is the output vector (or system's response vector of size $p \times 1$) and $\mathbf{v}(t)$ is the input or control vector (size $m \times 1$). We also assume that the system is controllable, that is for any initial state $\mathbf{x}_1$ and any final state $\mathbf{x}_2$ there exists an input signal $\mathbf{v}$ that transfers the system from state $\mathbf{x}_1$ to state $\mathbf{x}_2$ in a finite time. A neccessary and sufficient condition of controllability is $\mathrm{rank}\,[A, AB, \ldots, A^{n-1}B] = n$. Tracking a zero signal for system (2) requires introduction of a feedback control $\mathbf{v}$ (that is, $\mathbf{v}$ depends on the state $\mathbf{x}$), such that $\lim_{t\to\infty} \mathbf{y}(t) = 0$. Due to a well known theorem in control theory (the Bass-Gura Theorem, Rugh (1996)), controllability of system (2) implies existence of a matrix K, such that the eigenvalues of $A - BK$ have negative real parts (in fact, it is always possible to solve for K such that the eigenvalues of $A - BK$ are arbitrary). Then, the control law $\mathbf{v}(t) = -K\mathbf{x}(t)$ automatically solves this tracking problem. Indeed, the state of equation (2) implies $\dot{\mathbf{x}} = (A - BK)\mathbf{x}$, hence $\mathbf{x}(t) = \mathrm{e}^{(A-BK)t}\mathbf{x}_0$ and consequently, $\mathbf{y}(t) = C\mathrm{e}^{(A-BK)t}\mathbf{x}_0$. It is clear from the latter expression that $\lim_{t\to\infty} \mathbf{y}(t) = 0$.

Next, let us consider the problem of tracking a general target signal $\mathbf{r}(t)$ for system (2); that is, we require $\lim_{t\to\infty} (\mathbf{y}(t) - \mathbf{r}(t)) = 0$. We now assume that $p = m$ and that $(CB)^{-1}$ exists. We also do not require the system that is controllable. Define $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{r}(t)$ and let H be any Hurwitz matrix; that is a matrix whose eigenvalues have negative real parts. Then, an easy computation implies:

$$\dot{\mathbf{e}} = H\mathbf{e} + (CB\mathbf{v} + CA\mathbf{x} - \dot{\mathbf{r}} - H\mathbf{e}), \tag{3}$$

hence, if we introduce the feedback control

$$\mathbf{v} = (CB)^{-1}(\dot{\mathbf{r}} + H(C\mathbf{x} - \mathbf{r}) - CA\mathbf{x}) \tag{4}$$

equation (3) becomes $\dot{\mathbf{e}} = H\mathbf{e}$, which implies $\lim_{t\to\infty} (\mathbf{y}(t) - \mathbf{r}(t)) = 0$.

Finally, we present another solution of the tracking problem for a general target $\mathbf{r}$. The feedback control law we use is similar in form to the one which will be introduced later for the convection-diffusion equation, however, it is valid under rather restrictive hypotheses on matrices B and C. We assume that $p = n$ and that C^{-1} exists. Moreover, we require that system (2) be controllable. Then, the Bass-Gura Theorem implies existence of a matrix K, such that $A - BK$ has negative real part eigenvalues. Next, we introduce the control input

$$\mathbf{v} = B^{-1}(\dot{\mathbf{s}} - A\mathbf{s}) - K(\mathbf{x} - \mathbf{s}), \tag{5}$$

where $\mathbf{s} = C^{-1}\mathbf{r}$. Note that in the latter expression $\dot{\mathbf{s}} - A\mathbf{s}$ is the state space equation's operator evaluated at the scaled (by C^{-1}) target $\mathbf{r}$. The term $\mathbf{x} - \mathbf{s}$ is the (scaled by C^{-1}) control error $\mathbf{y} - \mathbf{r}$. Substituting equation (5) in the state equation (2) we infer

$$(\dot{\mathbf{x}} - \dot{\mathbf{s}}) = (A - BK)(\mathbf{x} - \mathbf{s}).$$

Clearly, $\mathbf{x}(t) - \mathbf{s}(t) \to 0$ as $t \to \infty$ (since $A - BK$ is a Hurwitz matrix), and therefore $\lim_{t\to\infty}(\mathbf{y}(t) - \mathbf{r}(t)) = 0$. The motivation for the definition of the control law applied in the present study for equation (1) is based on the idea that for large time t, it is desired that the fluid temperature field be very close to the profile T_*, so $\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - \alpha\nabla^2 T = g$ becomes (at least approximately):

$$\frac{\partial T_*}{\partial t} + \mathbf{u}\cdot\nabla T_* - \alpha\nabla^2 T_* = g \tag{6}$$

for large time t. Hence, it is reasonable to define a controller using the Ansatz

$$g = \frac{\partial T_*}{\partial t} + \mathbf{u}\cdot\nabla T_* - \alpha\nabla^2 T_* + G(T, T_*)$$

where the function $G(T,T_*)$ is such that $|G(T, T_*)|$ attains very small values when t is large, so that equation (6) is satisfied. The most obvious choice of $G(T,T_*)$ is $T - T_*$, possibly scaled by a factor β. It is clear that the magnitude of β will influence, how rapidly $|G(T, T_*)|$ will decrease and as a consequence how fast equation (6) is going to be valid.

In the present tracking problem, we use the weak formulation of equation (1) given below where L^2 and H^1, H^{-1} represent the usual Hilbert spaces (Evans, 1998).

Find $T \in L^2((0,t_*),\ H_0^1(\Omega)) \bigcap H^1((0,t_*),\ H^{-1}(\Omega))$, such that

$$\langle T_t, v\rangle + a(T,v) + b(v,T) = \langle g, v\rangle, \forall v \in H_0^1(\Omega) \tag{7}$$

$$T(\cdot,0) = T_0 \tag{8}$$

a.e in (0, t_*), where

$$a(T,v) = \alpha \int_\Omega \nabla T \cdot \nabla v \, \mathrm{d}\mathbf{x},$$

$$b(v,T) = \int_\Omega v\mathbf{u} \cdot \nabla T \, \mathrm{d}\mathbf{x},$$

with the following linear feedback control law:

$$\langle g, v\rangle = \left\langle \frac{\partial T_*}{\partial t}, v \right\rangle + a(T_*, v) + b(v, T_*) + \beta(T - T_*, v)_{\mathrm{L}^2(\Omega)} \tag{9}$$

where $T_* \in L^2((0,t_*),\ \ H_0^1(\Omega)) \bigcap H^1((0,t_*),\ \ H^{-1}(\Omega))$ is the target temperature profile and β is a constant that determines the controller's performance. In fact a value of β outside a certain range will result in divergence from the target temperature. From the first Gauss-Green identity, equation (9) implies

$$g = \frac{\partial T_*}{\partial t} + \mathbf{u} \cdot \nabla T_* - \alpha \nabla^2 T_* + \beta(T - T_*) \tag{10}$$

which can be interpreted physically as a heat source control term. The term $\beta(T - T_*)$ is clearly the scaled pointwise error between system's response and target temperature T_*. Furthermore, the control law (equation (10)) is quite similar to equation (5) for the LTI case; it is a linear combination of the problem's differential operator evaluated at the target T_* and the control error $T - T_*$.

Proposition 1. Let $\beta < \alpha C^2$, where α is the thermal diffusivity, C is the Poincaré constant, and $w = T - T^*$. Then

$$\|w\|_{L^2(\Omega)} \leq \mathrm{e}^{(\beta - \alpha C^2)t} \|w(\cdot,0)\|_{L^2(\Omega)}, \quad a.e. \ \text{in}\, (0,t_*) \tag{11}$$

For the special case $t_* = \infty$, the following asymptotic result holds; $L^2(\Omega) - \lim_{t\to\infty} w(\cdot,t) = 0$ (and consequently $L^2(\Omega) - \lim_{t\to\infty} (T(\cdot,t) - T_*(\cdot,t)) = 0$)

Proof. (See Kavouklis, 2002).

It follows that the feedback control law defined in equation (9) is indeed a solution of the tracking problem. Moreover, the temperature field converges exponentially in time to the target profile, as inequality (11) indicates. Clearly, the value of Poincaré's constant is crucial to the controllet's performance. An analogous result to that of Proposition 1 holds for the backward Euler discretized problem: □

Proposition 2. If $\beta < \alpha C^2$, then

$$\|w^{n+1}\|_{L^2(\Omega)} \leq \frac{1}{[1-(\beta-\alpha C^2)\Delta t]^{n+1}}\|w^0\|_{L^2(\Omega)}, \quad \forall n \in \{0,1,\ldots,M\},$$

and for the case $t_* = \infty$, $L^2(\Omega) - \lim_{t\to\infty} w^n = 0$ (and therefore $L^2(\Omega) - \lim_{n\to\infty}(T^n - T_*^n) = 0$)

Proof. (See Kavouklis, 2002).

Consider a finite element subspace S_h of $H_0^1(\Omega)$. The fully discrete formulation follows as:

find $\{T_h^{n+1}\}_{n\in\{0,1,\ldots,M\}} \in S_h$ such that

$$\begin{aligned} &\frac{1}{\Delta t}\left(T_h^{n+1}, v_h\right) + \alpha\left(T_h^{n+1}, v_h\right) + b\left(v_h, T_h^{n+1}\right) \\ &\quad = \langle g^{n+1}, v_h\rangle + \frac{1}{\Delta t}\left(T_h^n, v_h\right)_{L^2(\Omega)}, \forall v_h \in S_h \end{aligned} \tag{12}$$

$$T_h^0 = P_h(T_0)$$

where $g^{n+1} \in S_h^*$ and P_h is the elliptic projection. The control functional g^{n+1} is defined by

$$\begin{aligned} \langle g^{n+1}, v_h\rangle = {} & \frac{1}{\Delta t}\left(P_h\left(T_*^{n+1} - T_*^n\right), v_h\right)_{L^2(\Omega)} + a\left(P_h\left(T_*^{n+1}\right), v_h\right) \\ & + b\left(v_h, P_h\left(T_*^{n+1}\right)\right) + \beta\left(T_h^{n+1} - P_h\left(T_*^{n+1}\right), v_h\right)_{L^2(\Omega)} \end{aligned} \tag{13}$$

Substituting equation (13) into equation (12) and defining $w_h^{n+1} = T_h^{n+1} - P_h\left(T_*^{n+1}\right)$, we have equivalently:

find $\{w_h^{n+1}\}_{n\in\{0,1,\ldots,M\}} \in S_h$ such that

$$\begin{aligned} &\left(\frac{1}{\Delta t} - \beta\right)\left(w_h^{n+1}, v_h\right)_{L^2(\Omega)} + a\left(w_h^{n+1}, v_h\right) + b\left(v_h, w_h^{n+1}\right) = \\ &\frac{1}{\Delta t}\left(w_h^n, v_h\right)_{L^2(\Omega)}, \forall v_h \in S_h \end{aligned} \tag{14}$$

$$w_h^0 = T_h^0 - P_h\left(T_*^0\right) = P_h\left(T_0 - T_*^0\right) \tag{15}$$

and □

Proposition 3. If $\beta < \alpha C^2$, then

$$\|w_h^{n+1}\|_{L^2(\Omega)} \leq \frac{1}{(1-(\beta-\alpha C^2)\Delta t)^n}\|w_h^0\|_{L^2(\Omega)}, \quad n = 0, 1, \ldots, M \tag{16}$$

and $L^2(\Omega) - \lim_{n\to\infty} w_h^n = 0$ for the case $t_* = \infty$.

Proof. (See Kavouklis, 2002).

This leads directly to the following result: □

Theorem. Let $\beta < \alpha C^2$ and $T_* \in L^\infty((0,t_*), \; L^2(\Omega)) \bigcap L^2((0,t_*), H^2(\Omega))$. Then

$$\|T_h^{n+1} - T_*^{n+1}\|_{L^2(\Omega)} \leq \frac{1}{(1-(\beta-\alpha C^2)\Delta t)^n}\|w_h^0\|_{L^2(\Omega)} + h^2\|T_*^{n+1}\|_{L^2(\Omega)}, \quad n = 0, 1, \ldots, M$$

and therefore, $\lim_{n\to\infty}\lim_{h\to 0}\|T_h^{n+1} - T_*^{n+1}\|_{L^2(\Omega)} = 0$, if $t_* = \infty$.

Proof. We write

$$T_h^{n+1} - T_*^{n+1} = (T_h^{n+1} - P_h(T_*^{n+1})) + (P_h(T_*^{n+1}) - T_*^{n+1}).$$

Then

$$\|T_h^{n+1} - T_*^{n+1}\|_{L^2(\Omega)} \leq \|w_h^{n+1}\|_{L^2(\Omega)} + \|P_h(T_*^{n+1}) - T_*^{n+1}\|_{L^2(\Omega)}.$$

Since

$$\|P_h(T_*^{n+1}) - T_*^{n+1}\|_{L^2(\Omega)} \leq h^2\|T_*^{n+1}\|_{L^2(\Omega)} \tag{17}$$

we obtain

$$\|T_h^{n+1} - T_*^{n+1}\|_{L^2(\Omega)} \leq \frac{1}{(1-(\beta-\alpha C^2)\Delta t)^n}\|w_h^0\|_{L^2(\Omega)} + h^2\|T_*^{n+1}\|_{L^2(\Omega)}. \tag{18}$$

Using $\beta < \alpha C^2$ and as $\|T_*^{n+1}\|_{L^2(\Omega)}$ remains bounded, equation (18) implies $\lim_{n\to\infty}\lim_{h\to 0}\|T_h^{n+1} - T_*^{n+1}\|_{L^2(\Omega)} = 0$, for the case $t_* = \infty$, and the proof is completed. □

Finally, we note that the condition $T_* \in L^2((0,t_*), H^2(\Omega))$ in the hypotheses of the Theorem is true when $\partial\Omega \in C^2$. Moreover, this regularity condition implies the interpolation inequality (17) (Becker *et al.*, 1981).

Finite element algorithm

In the preceding Theorem, we show convergence of the approximate solution to the target temperature. However, in practice, instead of solving directly the actual control problems (12) and (13) for T_h^n, we find it more convenient to solve problem (14) for the error w_h^n and then reconstruct T_h^n from the identity $T_h^n = w_h^n + P_h(T_*^n)$. Let $\{\phi_1, \phi_2, \ldots, \phi_{M_h}\}$ be a basis of S_h. Allowing $v_h = \phi_i$, $i = 1, \ldots, M_h$ and introducing the expression for w_h, we obtain:

$$\sum_{j=1}^{M_h} \left(\left(\frac{1}{\Delta t} - \beta \right) (\phi_j, \phi_i)_{L^2(\Omega)} + a(\phi_j, \phi_i) + b^{n+1}(\phi_i, \phi_j) \right) c_j^{n+1}$$
$$= \frac{1}{\Delta t} \sum_{j=1}^{M_h} (\phi_j, \phi_i)_{L^2(\Omega)} c_j^n, \quad i = 1, \ldots, M_h. \tag{19}$$

or

$$U^{n+1} c^{n+1} = X c^n,$$

for the unknown coefficients c of w_h, where U^{n+1} and X are $M_h \times M_h$ matrices as indicated in equation (19). Moreover, c is obtained by elliptic projection (equation (15))

$$\sum_{j=1}^{M_h} (\phi_j, \phi_i)_{H_0^1(\Omega)} c_j^0 = (T_0 - T_*^0, \phi_i)_{H_0^1(\Omega)}, \quad i = 1, \ldots, M_h,$$

or in matrix form

$$X c^0 = \mathbf{r}^0 \tag{20}$$

The computational algorithm follows as:

(1) Set $n = 0$ and solve $Xc^0 = \mathbf{r}^0$ to obtain w_h^0,

(2) Compute the matrix U^{n+1},

(3) Solve $U^{n+1} c^{n+1} = X c^n$ to obtain w_h^{n+1},

(4) Solve $X c_*^{n+1} = \mathbf{r}_*^{n+1}$ where

$$T_*^{n+1} = \sum_{j=1}^{M_h} c_{*,j}^{n+1} \phi_j, \quad r_{*,j}^{n+1} = (T_*^{n+1}, \phi_j)_{H_0^1(\Omega)}, \quad j = 1, \ldots, M_h$$

to obtain the projection $P_\mathrm{h}(T_*^{n+1})$ of T_*^{n+1} on S_h.

(5) Compute T_h^{n+1} using the identity $T_h^{n+1} = w_h^{n+1} + P_h(T_*^{n+1})$,

(6) Set $n = n + 1$ and go to step 2.

Remark. To avoid oscillatory behavior of w_h, a local Peclet condition needs to be satisfied, hence h or $\|u\|$ must be sufficiently small.

Numerical studies

Consider the tracking problem on $\Omega = (0,1)^2$, using a uniform Cartesian grid of bilinear elements.

Lemma. For $\Omega = (0,1)^2$, and any $u \in H_0^1(\Omega)$ the Poincaré inequality holds with Poincaré constant $C = \frac{1}{\sqrt{2}}$. That is,

$$\frac{1}{\sqrt{2}}\|u\|_{H_0^1(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}$$

Proof. The proof (Kavouklis, 2002) is a consequence of the G-N-S inequality (e.g. Evans, 1998).

Now, let us consider four test cases – one for a steady state target profile and three cases with transient target temperature profiles. The incompressible velocity field we use is $\mathbf{u} = (u, v)$, where

$$u(x,y,t) = 2(1-\cos(2\pi tx))(1-x^2)[\pi t\sin(2\pi ty)(1-y^2) - y(1-\cos(2\pi ty))]$$

$$v(x,y,t) = 2(1-\cos(2\pi ty))(1-y^2)[\pi t\sin(2\pi tx)(1-x^2) - x(1-\cos(2\pi tx))]$$

The initial temperature field is a truncated Gauss bell function of amplitude $\frac{1}{2}$, centered at $\left(\frac{1}{4}, \frac{3}{4}\right)$, whose support has radius $\frac{1}{8}$; that is,

$$T(x,y,t) = \begin{cases} \frac{e}{2} e^{\frac{1}{64\left[\left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2-\frac{1}{64}\right]}}, & \text{if } \left(x-\frac{1}{4}\right)^2+\left(y-\frac{3}{4}\right)^2 < \frac{1}{64} \\ 0, & \text{otherwise} \end{cases}$$

The target temperature profiles are: □

Case 1. The same truncated Gauss bell of amplitude $\frac{1}{2}$, but now centered at $\left(\frac{3}{4}, \frac{1}{4}\right)$ as a steady target.

Case 2. The same form of Gauss bell, centered at $\left(\frac{3}{4}, \frac{1}{4}\right)$, but with amplitude d time variant (i.e. $d = \frac{1}{2} - \frac{1}{4}\sin(t)$.

Case 3. A similar Gauss bell to that in case 1 but with center moving on the circumference of the circle $\frac{1}{8}(5+\cos(2\pi t),\ 2+\sin(2\pi t))$.

Case 4. A moving Gauss bell, as in case 3, but with amplitude d changing in time (i.e. $d = \frac{1}{2} - \frac{1}{4}\sin(t)$) as in case 2.In Figures 1 and 2 we plot $\|T_h - T_*\|_{L^2(\Omega)}$ against time for all four cases, when the convergence condition $\beta < \alpha C^2$, $C = \frac{1}{\sqrt{2}}$, on the control constant is violated. The mesh size and thermal diffusivity are $h = 0.1$ and $\alpha = 0.1$, respectively, and the time step $\Delta t = 0.2$ for case 1 and $\Delta t = 0.1$ for cases 2-4. The value of β is 20, 50, 60 and 70 for cases 1-4, respectively. Clearly, the controller fails to track the target temperature field and in fact the error blows up relatively rapidly as the exponential nature of inequality (11) dictates. An objection that may be raised at this point is that the values of β we have chosen are much greater than the quantity αC^2 (which is equal to 0.05 for this case). However, for a value of

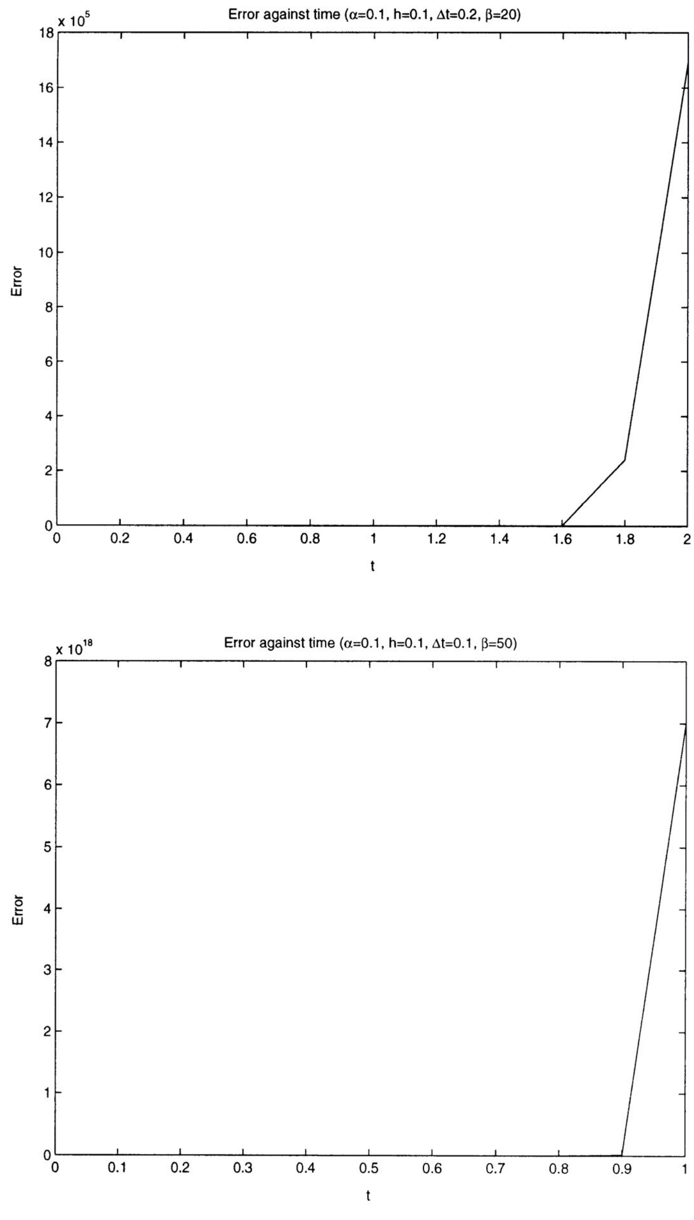
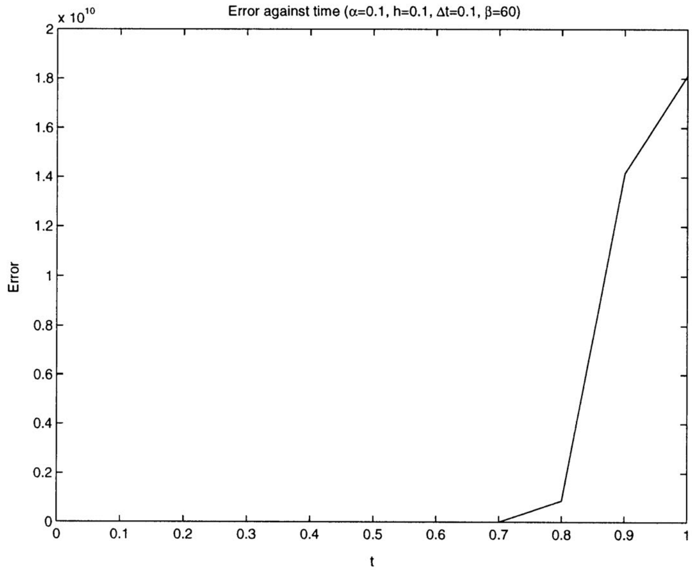


Figure 1.
L^2-error when $\beta \geq \alpha C^2$: cases 1 (left) and 2 (right)



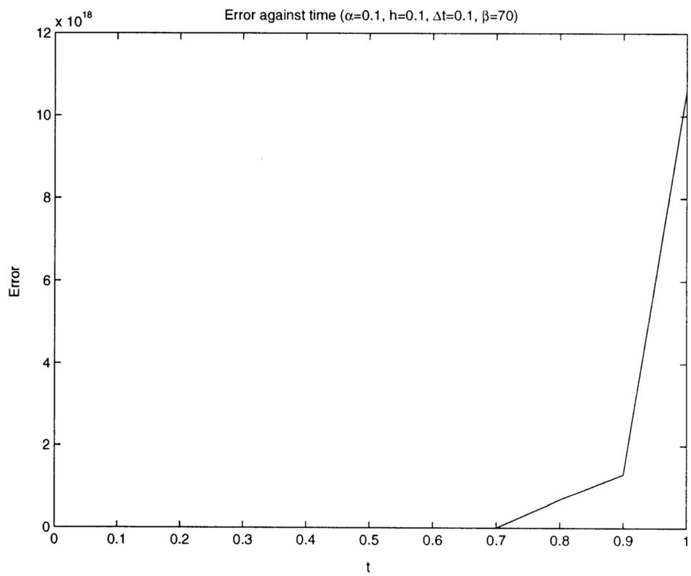


Figure 2.
L^2-error when $\beta \geq \alpha C^2$: cases 3 (left) and 4 (right)

β slightly greater than 0.05, the same singular behavior of the error is observed, but at a greater time t.

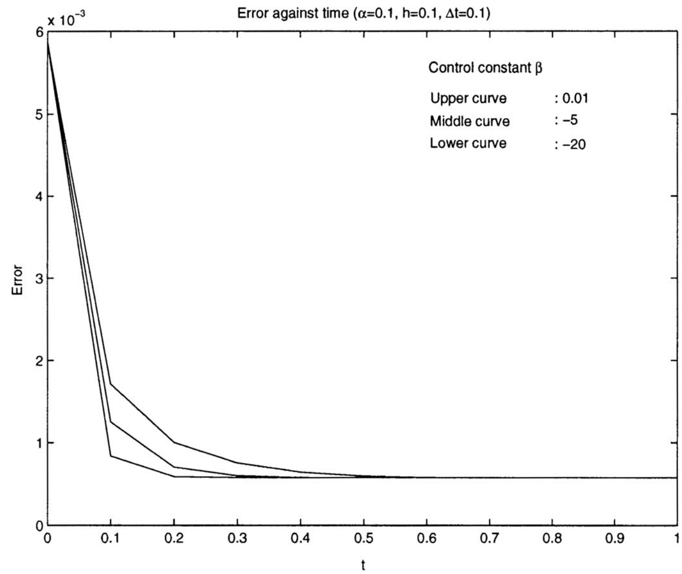
In Figures 3 and 4 we have plotted the L^2-error $\|T_h - T_*\|_{L^2(\Omega)}$ for all four cases and for different values of the control parameter β. The mesh size is $h = 0.1$ for cases 1 and 2 and $h = 0.03$ for cases 3 and 4. Thermal diffusivity $\alpha = 0.1$ is for all four cases. Here, the convergence criterion $\beta < \alpha C^2$ is satisfied and the error decreases as is expected from inequalities (11) and (18). We also observe that as β decreases, the controllet's performance is enhanced (i.e. the error curve falls more rapidly). This behavior is also in agreement with estimates (11) and (18). Also we observe that the error after being decreased rapidly attains a small asymptotic value, which in fact cannot be improved by reducing β. Clearly, this behavior is due to the term $h^2\|T_*^{n+1}\|_{L^2(\Omega)}$ in inequality (18), which dominates the error; hence a finer mesh has to be selected in order to improve our numerical controllet's performance. Indeed, as is evident from Figure 4 (compared to Figure 3), reduction of the mesh size results in a smaller limit value for the L^2-error.

Figures 5 and 6 shows quantitatively the speed of the controller in terms of the number of steps required to achieve an error level less than 0.001. For this case study, we used a mesh size $h = 0.1$ and a time step $\Delta t = 0.01$. The thermal diffusivity α was taken as 0.1. The range of control constant β we considered was from -10 to 0. Clearly, as β increases, a larger number of time steps is needed to obtain an error less than 0.001. This is expected from estimates (11) and (18) and is of course in agreement with the qualitative behavior of the error related to β in Figures 3 and 4.

In the remaining Figures 7-9, we give plots of both the computed controlled and target temperature fields for case 4, with $h = \frac{1}{60}$, $\Delta t = 0.1$, $\beta = 0.01$, $\alpha = 0.1$ and $t \in [0, 1]$. It is clear from these illustrations that the controller decreases the initial Gauss's bell amplitude while simultaneously adjusting fluid temperature to the target profile. By time $t = 0.2$, the initial Gauss bell has almost faded away and after that the controller drives the system to its desired state. The choice of small enough mesh size so that the local Peclet condition is satisfied is important here. Had we used a larger mesh size, the controller would have been able to adjust the location of the computed bell to that of the target, however the computed amplitude would have not been accurate, due to the presence of oscillations of the numerical solution.

Conclusions

In this study, we consider the approach in Gunzburger and Manservisi (2000a) and develop it for control of convection diffusion. Theoretical convergence results for the controllers are proved and the strategy is implemented and tested for several 2D cases. Numerical studies are carried out for both steady state and transient problems.



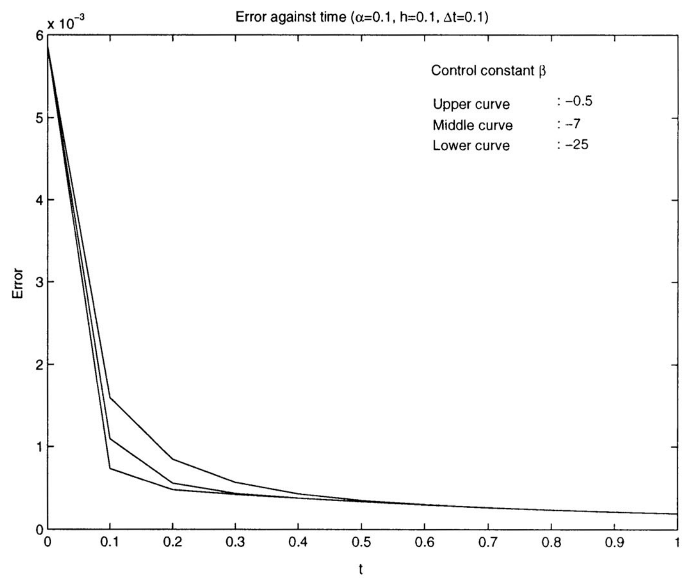


Figure 3.
L^2-error when $\beta < \alpha C^2$: cases 1 (left) and 2 (right)

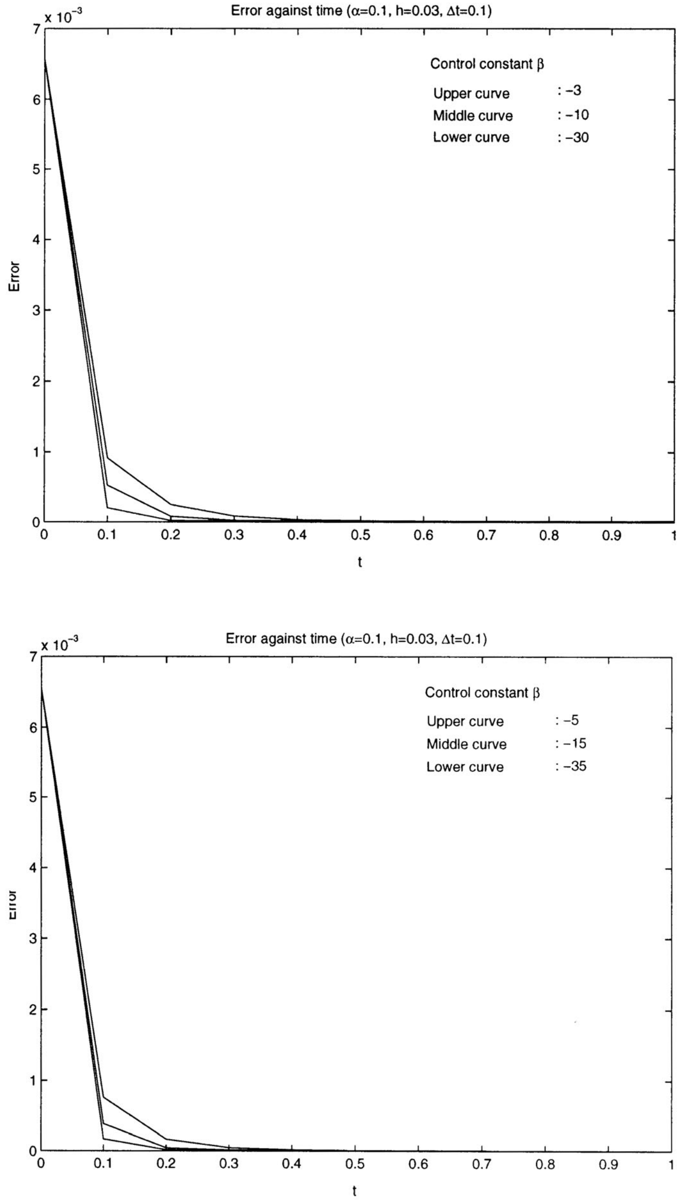


Figure 4.
L^2-error when $\beta < \alpha C^2$: cases 3 (left) and 4 (right)

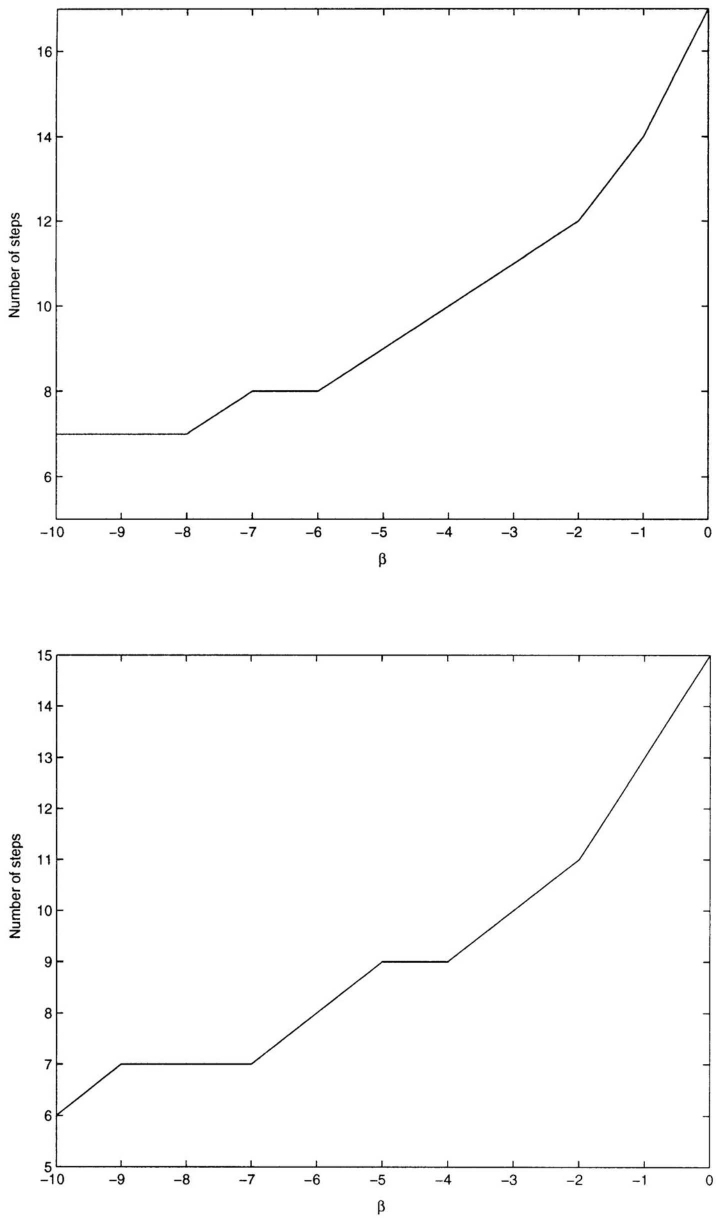
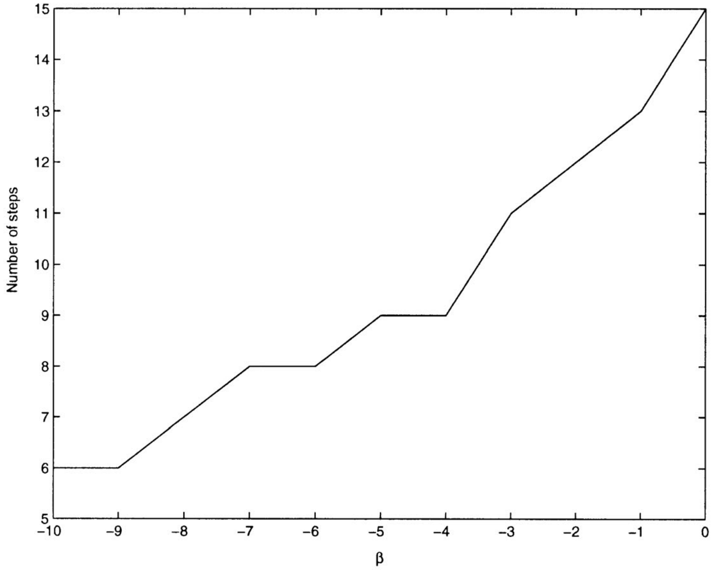


Figure 5.
Number of steps required for L^2-error less than 0.001 against control constant β: cases 1 (left) and 2 (right)



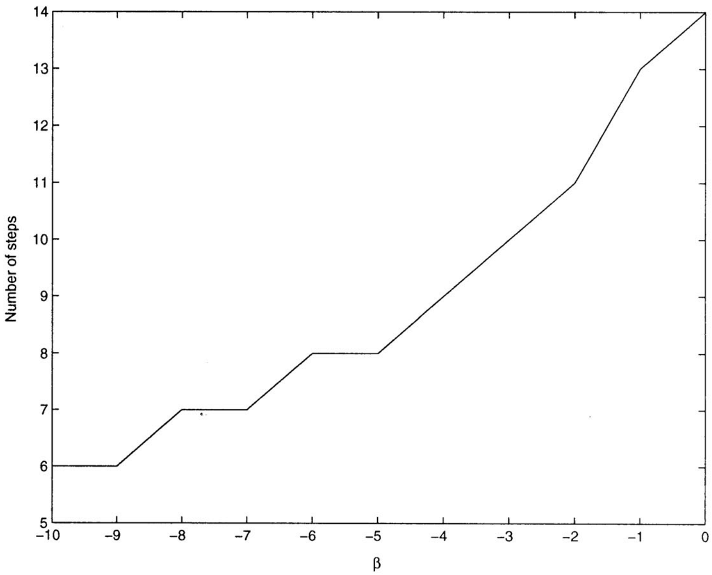


Figure 6.
Number of steps required for L^2-error less than 0.001 against control constant β: cases 3 (left) and 4 (right)

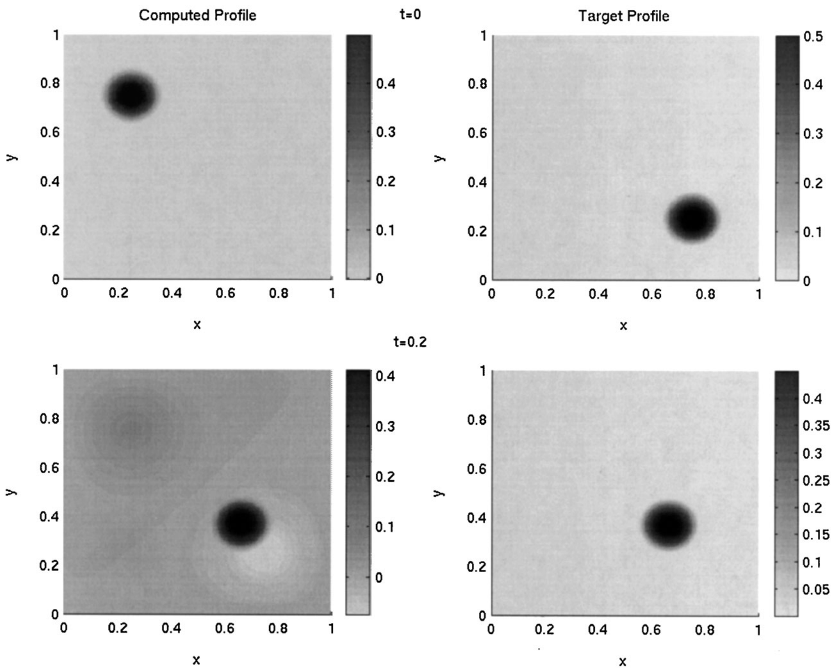


Figure 7.
Computed and target profiles for case 4, $t = 0$, 0.2, $\Delta t = 0.1$, $h = 1/60$, $\alpha = 0.1$, $\beta = 0.01$

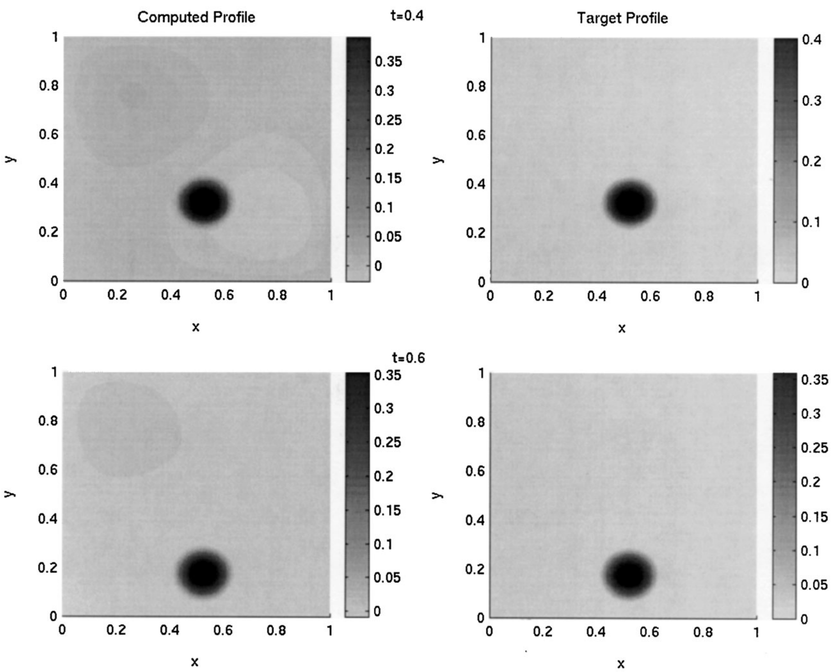


Figure 8.
Computed and target profiles for case 4, $t = 0.4$, 0.6, $\Delta t = 0.1$, $h = 1/60$, $\alpha = 0.1$, $\beta = 0.01$

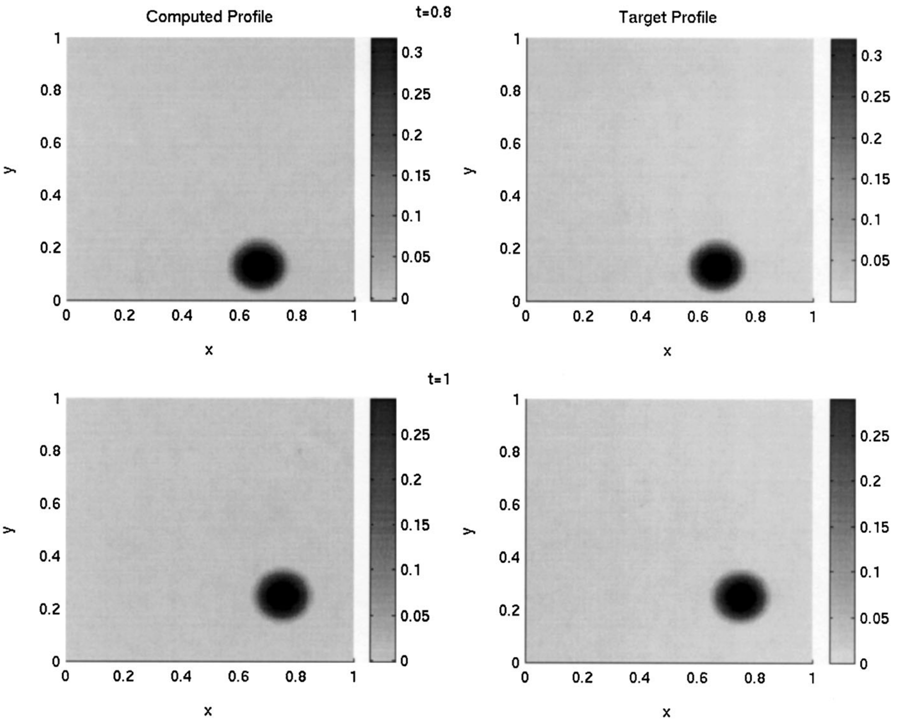


Figure 9. Computed and target profiles for case 4, $t = 0.8, 1.0$, $\Delta t = 0.1$, $h = 1/60$, $\alpha = 0.1$, $\beta = 0.01$

We remark that the approach may be particularly, useful in loosely coupled transport applications where the convective velocity is first computed and then the subsequent heat or mass transfer problem controlled as suggested here. If there is strong feedback between the transport problem through, for instance, buoyancy, thermo-capillary surface tension or density gradients, then the control problem is more complex since the velocity $\mathbf{u}$ is no longer known a priori but instead evolves in a coupled way via the momentum equations and analyzing this problem remains an open question.

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